Year 13 Mathematics

EAS 3.5

Complex Numbers

Robert Lakeland & Carl Nugent

Contents

NuLake Ltd
Innovative Publisher of Mathematics Texts

Complex Numbers 3.5

This achievement standard involves applying the algebra of complex numbers in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply the algebra of complex numbers in solving problems.	• Apply the algebra of complex numbers, using relational thinking, in solving problems.	• Apply the algebra of complex numbers, using extended abstract thinking, in solving problems.

- This achievement standard is derived from Level 8 of The New Zealand Curriculum and is related to the achievement objectives:
  - manipulate complex numbers and present them graphically
  - form and use polynomial, and other non-linear equations

  in the Mathematics strand of the Mathematics and Statistics Learning Area.

- Apply the algebra of complex numbers in solving problems involves:
  - selecting and using methods
  - demonstrating knowledge of concepts and terms
  - communicating using appropriate representations.

- Relational thinking involves one or more of:
  - selecting and carrying out a logical sequence of steps
  - connecting different concepts or representations
  - demonstrating understanding of concepts
  - forming and using a model;

  and relating findings to a context, or communicating thinking using appropriate mathematical statements.

- Extended abstract thinking involves one or more of:
  - devising a strategy to investigate or solve a problem
  - identifying relevant concepts in context
  - developing a chain of logical reasoning, or proof
  - forming a generalisation;

  and also using correct mathematical statements, or communicating mathematical insight.

- Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.

- Methods include a selection from those related to:
  - quadratic and cubic equations with complex roots
  - Argand diagrams
  - polar and rectangular forms
  - manipulation of surds
  - manipulation of complex numbers
  - loci
  - De Moivre's theorem
  - equations of the form $z^n = r \text{ cis } \theta$, or $z^n = a + bi$ where a and b are real and n is a positive integer.

Solving Quadratic Equations by Completing the Square

Completing the Square

Not all quadratics factorise. Therefore we need a technique that will enable us to solve quadratics that do not factorise. The first technique we look at is called **completing the square.**

Consider the pattern

$$(x \pm a)^2 = x^2 \pm 2ax + a^2$$

This expression is called a perfect square and is the basis of completing the square.

The method requires us to rewrite the quadratic as a perfect square adjusting the constant term to match the given quadratic.

We then take the square root of both sides of the expression and make x the subject to obtain the solution(s).

Solving a quadratic by completing the square is the easiest method when you are expected to express the answer in surd form.

Completing the Square is the way to go if you want an exact answer.

The square of half the coefficient of the x term will complete a perfect square.

To find the constant when completing the square, square the 'a' of $(x \pm a)^2$ and add or subtract the required amount.

Example

Use completing the square to solve the quadratic equation

$$x^2 - 6x + 2 = 0$$

Put the constant on the right-hand side

$$x^2 - 6x = {}^-2$$

Complete the square with $a = {}^-3$ (half the $^-6$).

$$x^2 - 6x + (^-3)^2 = {}^-2 + (^-3)^2$$

$$x^2 - 6x + 9 = {}^-2 + 9$$

$$(x - 3)^2 = {}^-2 + 9$$

$$(x - 3)^2 = 7$$

Solve

$$x - 3 = \pm\sqrt{7}$$

$$x = 3 \pm \sqrt{7}$$

$$x = 3 + \sqrt{7} \text{ and } x = 3 - \sqrt{7}$$

Example

Use completing the square to solve the quadratic equation

$$2x^2 + 4x - 8 = 0$$

Put the constant on the right-hand side

$$2x^2 + 4x = 8$$

Common factor 2

$$2(x^2 + 2x) = 8$$

Divide both sides by 2

$$x^2 + 2x = 4$$

Complete the square with a = 1 (half of 2)

$$x^2 + 2x + (1)^2 = 4 + (1)^2$$

$$(x + 1)^2 = 4 + 1$$

$$(x + 1)^2 = 5$$

Solve

$$x + 1 = \pm\sqrt{5}$$

$$x = {}^-1 \pm \sqrt{5}$$

$$x = {}^-1 + \sqrt{5} \text{ and } x = {}^-1 - \sqrt{5}$$

The Remainder and Factor Theorems

Remainder Theorem

Consider the polynomial defined by

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

The degree of a polynomial is the highest power of x in the expression, in this case 'n', and its constant term is a_0.

If we have a polynomial p(x) and divide it by (x – a) to give a quotient Q(x) and remainder R, then we can write

$$p(x) = (x - a).Q(x) + R$$

If p(x) is divided by (x – a) and a remainder is obtained, then that remainder R is p(a), i.e. p(a) = R.

Factor Theorem

A factor is a divisor that gives a remainder of zero. We can use the Factor Theorem which is just a corollary to the Remainder Theorem to factorise more complicated polynomials.

If (x – a) is a factor of a polynomial p(x), then p(a) = 0. (i.e. the remainder will be 0).

Also if p(a) = 0 then (x – a) is a factor of p(x).

Example

Find the remainder when $p(x) = x^2 - 4x + 5$ is divided by (x – 1).

Given $p(x) = x^2 - 4x + 5$

Remainder = p(1)

$p(1) = 1^2 - 4(1) + 5$

$p(1) = 2$

So when p(x) is divided by (x – 1) the remainder is 2.

Extra Examples

Extra Notes

The proof of the remainder theorem could be required for Excellence.

Given	$p(x) = (x - a).Q(x) + R$
Letting x = a	$p(a) = (a - a).Q(a) + R$
therefore	$p(a) = 0.Q(a) + R$
thus	$p(a) = R$

The proof of the factor theorem could be required for Excellence.

Given $p(x) = (x - a).Q(x) + R$

If (x – a) is a factor of a polynomial p(x) then the remainder R will be 0.

	$p(x) = (x - a).Q(x)$
If x = a then	$p(a) = (a - a).Q(a)$
so	$p(a) = 0.Q(a)$
thus	$p(a) = 0$

Example

Find the remainder when $p(x) = 2x^2 + 6x - 3$ is divided by (2x – 1).

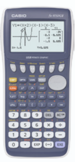
Given $p(x) = 2x^2 + 6x - 3$

Remainder = $p\left(\frac{1}{2}\right)$

$$p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) - 3$$

$$p\left(\frac{1}{2}\right) = \frac{1}{2}$$

So when p(x) is divided by (2x – 1) the remainder is $\frac{1}{2}$.



On the Casio 9750GII (or any graphics calculator) you can store an x value in memory.

e.g. [1] [ab/c] [2] [→] [x,ø,T] [EXE]

Then type in the required equation, i.e.

[2] [x,ø,T] [x²] [+] [6] [x,ø,T] [–] [3] [EXE] **and you get $\frac{1}{2}$, which is the remainder.**

Achievement – Answer the following questions.

126. For the polynomial $p(x) = 6x^3 - 13x^2 + 4$

a) Find the remainder when $p(x)$ is divided by $(x + 3)$.

b) Show that $(x - 2)$ is a factor of $p(x)$.

c) Find the three solutions of the equation $p(x) = 0$.

127. For $p(x) = x^3 - x^2 - 7x - 2$ find

a) the remainder when $p(x)$ is divided by $(x + 1)$.

b) the only rational factor.

128. Consider the polynomial $p(x) = x^3 + 4x^2 + 8x + 5$

a) Show that $(x + 1)$ is a factor of $p(x)$.

b) Write $p(x)$ in the form $(x + 1)(ax^2 + bx + c)$.

129. A polynomial $p(x) = 2x^3 - x^2 - 13x - k$

a) Find the remainder when $p(x)$ is divided by $(x - 3)$.

b) Find the value of k if $p(x)$ is divisible by $(x + 2)$.

c) Using the value of k obtained in b) above factorise the polynomial $p(x)$ completely.

130. Consider $p(x) = x^3 + ax^2 + bx - 12$. Given that $(x + 3)$ and $(x - 4)$ are factors of $p(x)$, factorise $p(x)$ completely.

131. The polynomial $p(x) = 2x^3 - ax^2 + bx + 48$ has $(x - 4)$ as a repeated factor, find the values of a and b.

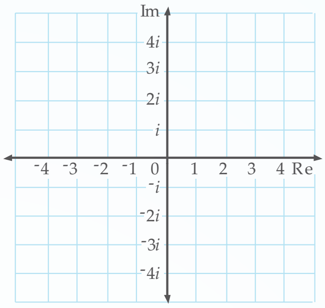
243. If $z = 3 \text{ cis } \frac{^-\pi}{2}$ and $w = 2 \text{ cis } \frac{3\pi}{4}$

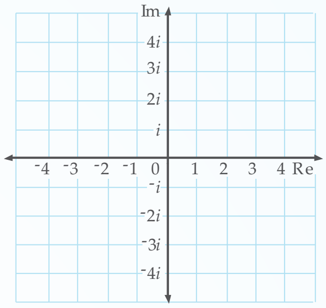
a) plot z and w on the first of the Argand diagrams.

b) find zw in polar form.

c) find $\frac{w^2}{z}$ in polar form.

d) plot your answers to b) and c) above on the second Argand diagram.





244. Two impedances $z_1 = 4 + 3i$ and $z_2 = 3 - 4i$ are connected in parallel in an electrical circuit. The combined effect of impedances connected in parallel is given by

$$\frac{1}{z_{combined}} = \frac{1}{z_z} + \frac{1}{z_2}$$

a) Find $\frac{1}{z_{combined}}$ and $z_{combined}$ in the form a + b*i*.

b) Convert $z_{combined}$ to polar form.

c) Given that V = 130 cis 0.145 volts use

$I = \frac{V}{z_{combined}}$ to find the current I in polar form.

245. In a simple electrical circuit there are three basic things to consider

- flow of electric current (I).
- resistance to that flow, called impedance (Z).
- electromotive force, called voltage (V).

All three are related by the formula V = I x Z which can be expressed in several ways

i.e. $V = I \times Z$ or $\frac{V}{Z} = I$ or $\frac{V}{I} = Z$

a) Compute the voltage, V (in polar form) when I = 5 cis 0.6435 and Z = 32.35 cis 1.052.

b) Compute the impedance, Z (in polar form) when V = 111.8 cis 2.034 and I = 6.325 cis $^-$2.820.

246. If $z = \frac{^-1 + \sqrt{3}i}{2}$

a) convert z to polar form.

b) find z^3 in polar form and then show $z^3 - 1 = 0$.

247. If two impedances $z_1 = 2 + 15i$ and $z_2 = 7 - 2i$ are connected in series to a supply voltage V of 240 volts and the current I (in amperes) is given $I = \frac{V}{Z}$ where $Z = z_1 + z_2$, find I in polar form.

Example

For $|z - 1| = |z - 3|$ describe the locus of z in the complex plane and obtain a Cartesian equation for the locus.

Solution

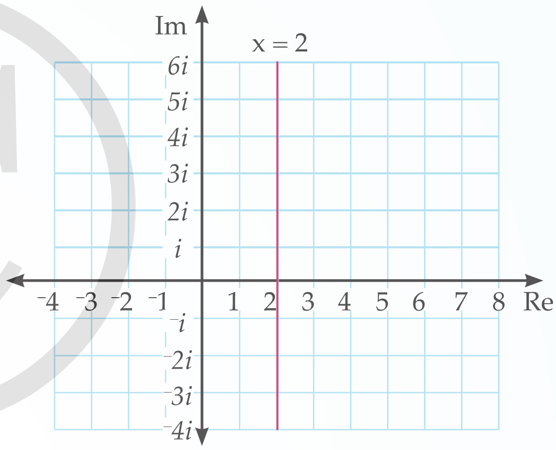
$$|z - 1| = |z - 3|$$

Substituting $z = x + yi$

we write

$$|(x + yi) - 1| = |(x + yi) - 3|$$
$$|(x - 1) + yi| = |(x - 3) + yi|$$
$$\sqrt{(x-1)^2 + y^2} = \sqrt{(x-3)^2 + y^2}$$
$$(x - 1)^2 + y^2 = (x - 3)^2 + y^2$$
$$x^2 - 2x + 1 + y^2 = x^2 - 6x + 9 + y^2$$
$$^-2x + 1 = {}^-6x + 9$$
$$4x = 8$$
$$x = 2$$

The locus of z is a vertical line through 2.



Example

For $|z| + |z - 2| = 6$ describe the locus of z geometrically and obtain a Cartesian equation for the locus.

Solution

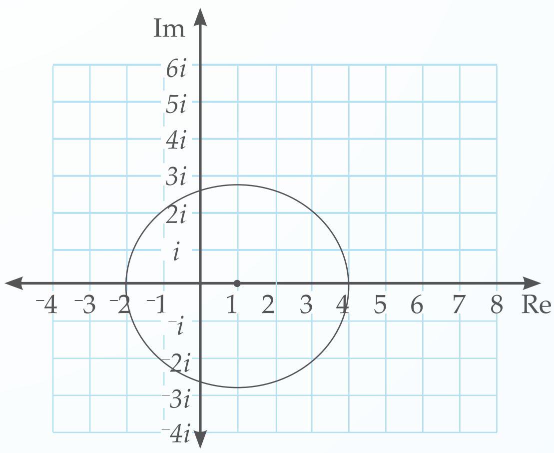
$$|z| + |z - 2| = 6$$

Substituting $z = x + yi$

we write

$$|(x + yi)| + |(x + yi) - 2| = 6$$
$$|(x + yi)| + |(x - 2) + yi| = 6$$
$$\sqrt{x^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 6$$
$$(x - 2)^2 + y^2 = (6 - \sqrt{x^2 + y^2})^2$$
$$x^2 - 4x + 4 + y^2 = 36 - 12\sqrt{x^2 + y^2} + x^2 + y^2$$
$$^-4x - 32 = {}^-12\sqrt{x^2 + y^2}$$
$$(^-4x - 32)^2 = 144(x^2 + y^2)$$
$$16x^2 + 256x + 1024 = 144x^2 + 144y^2$$
$$128x^2 - 256x + 144y^2 = 1024$$
$$128(x^2 - 2x) + 144y^2 = 1024$$
$$128(x - 1)^2 + 144y^2 = 1152$$
$$\frac{(x-1)^2}{9} + \frac{y^2}{8} = 1$$

The locus of z is an ellipse centre (1, 0) with major axis 6 and minor axis $2\sqrt{8}$.



Page 12 cont...

65. $2(k+2)^2 - 1 = 0$

$k = {}^-2 \pm \sqrt{\frac{1}{2}}$ or ${}^-2 \pm \frac{1}{\sqrt{2}}$

66. $x = {}^-1 \pm \sqrt{1 + \frac{k}{2}}$

67. $x = 2 \pm \sqrt{4 - \frac{k}{2}}$

68. $x = {}^-1 \pm \sqrt{1 + \frac{6}{k}}$

69. $x = \frac{{}^-1 \pm \sqrt{7}}{k}$

Page 14

70. $x = {}^-0.146, {}^-6.854$

71. $x = 1.854, {}^-4.854$

72. $x = 5, {}^-6$

73. $x = {}^-1, 2.5$

Page 15

74. $x = 1.143, 0.180$

75. $x = 3.886, {}^-0.886$

76. $x = {}^-3 \pm \sqrt{10}$

77. $x = 3 \pm \sqrt{6}$

78. $x = \frac{{}^-5 \pm \sqrt{29}}{2}$

79. $x = 2 \pm \sqrt{14}$

80. $x = {}^-4 \pm \sqrt{16 + k}$

81. $x = \frac{{}^-2 \pm \sqrt{13}}{k}$

82. $x = \frac{(k+2) \pm k}{2}$ or $k + 1, 1$

83. $x = 5k \pm \sqrt{26k}$

Page 17

84. $\Delta = 89$. Roots are **unequal**, **real** and **irrational**.

85. $\Delta = 25$. Roots are **unequal**, **real** and **rational**.

86. $\Delta = 0$. Roots are **equal** and **real**.

87. $\Delta = {}^-23$. Roots are **unequal** and **complex**.

88. $4 - 12c \geq 0$ so $c \leq \frac{1}{3}$. Includes equal as equal roots are real.

89. $4 + 16d < 0$ so $d < {}^-\frac{1}{4}$.

Page 17 cont...

90. $e^2 - 144 = 0$ so $e = \pm 12$.

91. $f^2 - 8 < 0$ so ${}^-\sqrt{8} < f < \sqrt{8}$.

92. $9k^2 - 32k < 0$ so

$k(9k - 32) < 0$

$0 < k < \frac{32}{9}$

93. $9k^2 - 60k + 96 < 0$ so

$(3k - 8)(k - 4) < 0$

$\frac{8}{3} < k < 4$

Page 19

94. $p({}^-1) = {}^-3$

95. $p(2) = 15$

96. $p({}^-1) = 3$

97. $p({}^-0.5) = {}^-3.4375$

98. $p({}^-2) = {}^-35$

99. $p(\frac{1}{3}) = 3.691$

100. $p(3) = 27 + 63 - 18 - 72 = 0$

101. $k = 22$

Page 20

102. $p({}^-1.5) = 2({}^-1.5)^3 + 9({}^-1.5)^2 - 1.5 - 12 = 0$

hence $(2x + 3)$ is a factor.

103. $k = 7$

104. $k = 12.5$

105. $k = 3, {}^-6$

106. $k = 19$

107. $q = 2$

108. $p(2a) = 16a^3 - 4a^3 - 6a^3 - 6a^3$

$p(2a) = 0$ hence a factor

109. $m = 2, n = 5$

110. $a = 3, b = {}^-7$

111. $a = 1, b = {}^-8$

Page 23

112. $(x + 1)(2x - 1)(2x + 1)$

$x = {}^-1, 0.5, {}^-0.5$

113. $(x - 4)(x - 2)(x + 1)$

$x = 4, 2, {}^-1$

114. $(x + 1)(2x + 1)(3x - 2)$

$x = 0.667, {}^-1, {}^-0.5$

115. $(x - 1)(2x - 5)(2x + 3)$

$x = {}^-1.5, 2.5, 1$

Page 24

116. $(x + 3)(x - 3)(4x - 1)$

$x = {}^-3, 3, 0.25$

117. $(x - 2)(x + 4)(2x - 1)$

$x = 2, {}^-4, 0.5$

118. $(x - 4)(3x - 1)(2x + 3)$

$x = 4, 0.333, {}^-1.5$

119. $(2x + 1)(2x - 1)(x - 1)$

$x = {}^-0.5, 0.5, 1$

120. $(3x - 1)(x - 2)(5x - 1)$

$x = 0.333, 2, 0.2$

121. $(x + 4)(x + 2)(x - 6)$

$x = {}^-4, {}^-2, 6$

122. $(3x - 1)(4x - 3)(2x - 3)$

$x = 0.333, 0.75, 1.5$

123. $(x - 3)(2x - 5)(3x - 2)$

$x = 3, 2.5, 0.667$

124. $(2x - 1)^2(2 - x)$

$x = 0.5, 2$

125. $({}^-3x - 2)^3$

$x = {}^-0.667$

Page 25

126. a) $p({}^-3) = {}^-275$

b) $p(2) = 48 - 52 + 4 = 0$

c) $x = \frac{2}{3}, \frac{{}^-1}{2}, 2$

127. a) $p({}^-1) = 3$

b) $(x + 2)$

128. a) $p({}^-1) = {}^-1 + 4 - 8 + 5 = 0$

b) $p(x) = (x + 1)(x^2 + 3x + 5)$

129. a) $(6 - k)$

b) $k = 6$

c) $(x + 2)(2x + 1)(x - 3)$

130. $p(x) = (x + 3)(x - 4)(x + 1)$

131. $a = 13, b = 8$

Page 28

132. $x = 1$

133. $x = 3$

134. $x = 2$

135. $x = 8$

136. $x = 6, 5$

137. $x = 16$

138. $x = 9$

139. $x = 0.333$